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Disentangling longitudinal and shear elastic waves by neo-Hookean soft devices

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Longitudinal and shear elastic waves are often spatially entangled in solid media and are difficult to be separated from each other. An efficient approach is proposed to physically split the two types of elastic waves by using soft hyperelastic materials. Invoking the hyperelastic transformation theory, we demonstrate that the longitudinal and shear elastic waves possess distinct characteristics and propagate in different paths in a deformed neo-Hookean material. This principle enables us to design tunable, broadband, and lossless wave-mode splitters by using a simple-shear-deformed neo-Hookean solid. Both theoretical analysis and numerical simulations confirm the high performance of the proposed soft device.
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Elastic waves have many applications ranging from seismology, nondestructive testing, telecommunications, and metallurgy to biomedical imaging. Usually, the longitudinal wave (primary wave or P-wave) and the shear wave (secondary wave or S-wave) simultaneously exist in a solid medium. The two types of elastic waves exhibit distinct characteristics and therefore have been used for different purposes. In shear wave elastography,[1] for example, the S-wave is used as an excitation, while the P-wave serves as an "eye" to monitor the deformation induced by the excitation. Although P- and S-waves travel with different wave speeds and thus exist independently in the time domain, they usually entangle with each other in space. The separation of the two types of elastic waves is of paramount significance for some technologically important applications. In seismic survey, for instance, a pure wave propagation mode is of great demand and has long been pursued to retrieve the information of earthquakes or geologic structures. In the literature, the so-called wave-mode or wave-field separation, in fact, refers to a datum processing technique.[2,3] Physical separations of the two types of elastic waves have long been a challenging issue.

An interface between two dissimilar solid media can separate the elastic waves of different modes because of their different refraction angles. In this fashion, however, wave-mode conversion[4] is apt to take place and thus incurs serious interference to the wave-mode separation. In addition, the converted wave mode may lead to a significant loss of energy. Therefore, the transformation theory[5] has been used to manipulate wave fields by bulk materials rather than interfaces. This theory has promoted the design of some wave control devices, e.g., invisibility cloaks[6,7] and wave absorbers,[8] in different areas of physics. On the basis of the form-invariant property of the governing equations of elastic waves, the transformation theory suggests that one can design a material capable of capturing the geometric transformation between the virtual space and the physical space. Elastodynamic metamaterials,[9,10] which assemble multiple individual elements, have been proposed as a candidate to achieve such extreme material characteristics as negative bulk modulus and mass density.[11] To date, however, the design and fabrication of metamaterials used for transformation devices remain a big challenge.

Recently, hyperelastic materials have been proposed to manipulate elastic waves.[12] Using the hyperelastic transformation theory, Parnell *et al.*[13,14] proved that anti-plane S-waves can be controlled by a pre-stressed incompressible neo-Hookean solid. Norris and Parnell[12] theoretically demonstrated that both P- and S-waves can be modulated by using semi-linear materials. All these works reveal that hyperelastic materials can behave like smart metamaterials[15] and realize some desired properties by tuning their deformation.

In both traditional elastodynamic transformation theory and hyperelastic transformation theory, the P- and S-waves are pursued to be manipulated in the same manner, as shown in Fig. 1(a). Apparently, this method cannot separate the elastic waves of different modes. In this letter, it is reported that a deformed neo-Hookean material can behave like an elastodynamic smart metamaterial for manipulating both in-plane and anti-plane S-waves. Considering the distinct characteristics of P- and S-waves, we propose a wave-mode splitting device that makes use of a simple-sheared neo-Hookean material. Both theoretical analysis and numerical simulations demonstrate the efficiency of such a soft device to disentangle P- and S-waves.

According to the hyperelastic transformation theory, one needs to seek an analogy between the pushing forward operation in the small-on-large theory[16] and the asymmetric transformation relation[17] in the traditional elastodynamic transformation theory. As discussed in the supplementary material,[18] a sufficient condition for the existence of such an analogy is

$$A_{ijkl} = \frac{\partial^2 W}{\partial F_{ji} \partial F_{lk}} = \text{constant}, \tag{1}$$

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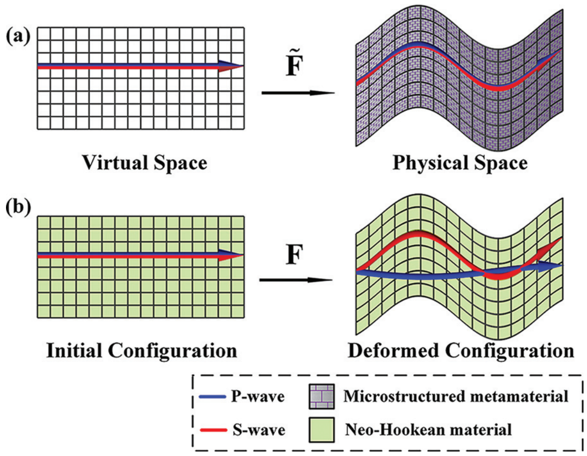

FIG. 1. Comparison between (a) the traditional transformation method and (b) the neo-Hookean transformation method. In (a), a spatial mapping denoted by $\tilde{\mathbf{F}}$ is constructed between the virtual space and the physical space. Both P- and S-waves are controlled by the same transformation $\tilde{\mathbf{F}}$, and a microstructured metamaterial is required to realize such a wave-control device. In (b), the wave-mode separation function is fulfilled by a deformed neo-Hookean material denoted by the deformation gradient tensor $\mathbf{F}$. In the deformed configuration, the S-wave path propagates along the material coordinate curves while the P-wave path does not.

where A_{ijkl} are the components of the fourth-order elastic tensor expressed in the initial configuration, W denotes the strain energy function of a hyperelastic solid and $F_{ij} = \partial x_i/\partial X_j$ denotes the deformation gradient, where X_i and x_i are the coordinates in the initial and the current configurations, respectively.

For a compressible isotropic neo-Hookean hyperelastic material, the strain energy density function can be expressed as[19,20]

$$W = \frac{\lambda}{2}(J-1)^2 - \mu \ln J + \frac{\mu}{2}(I_1 - 3)\,, \tag{2}$$

where $J = \det(\mathbf{F})$ is the volumetric ratio, $I_1 = \mathrm{tr}(\mathbf{B})$ is the first invariant of the right Cauchy-Green tensor $\mathbf{B} = \mathbf{F}^T \cdot \mathbf{F}$, while λ and μ are the first and second Lamé constants at the ground state, respectively. It is noticed that the first two terms of W in Eq. (2) are related only to the volumetric ratio J but not to the shear strains. Therefore, the strain energy density can be decomposed as $W = W^b + W^h$, where

$$W^b = \frac{\lambda}{2}(J-1)^2 - \mu \ln J\,, \qquad W^h = \frac{\mu}{2}(I_1 - 3). \tag{3}$$

In this fashion, the propagation of S-waves in a neo-Hookean hyperelastic material is governed only by the hybrid term W^h, whereas the propagation of P-waves depends on both the bulk term W^b and hybrid term W^h.

We first consider the effect of deformation on S-wave propagation in a neo-Hookean material. As discussed above, W^h is the only term that governs S-waves. One can easily prove that $A^h_{ijkl} = \partial^2 W^h/\partial F_{ji}\partial F_{lk} = \mu \delta_{ik}\delta_{jl}$ obey Eq. (1), where δ_{ij} is the Kronecker delta. For S-waves, therefore, there exists an analogy between the pushing forward relation and the asymmetric transformation relation. This indicates that the paths of S-waves propagating in a neo-Hookean solid will always attach in the material coordinates and deform along with the material deformation. Both in-plane and anti-plane S-waves can be manipulated by a deformed neo-Hookean material, and the material compressibility does not interfere with this prominent property.

Different from W^h, however, W^b does not satisfy Eq. (1), indicating that the deformation in a neo-Hookean material has a distinguishing influence on the travelling of P-wave. The above different propagation behaviors of P- and S-waves in a deformed neo-Hookean material are illustrated in Fig. 1(b). This observation inspires us to envision an efficient route to disentangling P- and S-waves with a device containing a deformed neo-Hookean material.

To illustrate the proposed wave-mode separation method, for example, we consider the propagation of an S-wave and a P-wave in a neo-Hookean material in a simple-shear deformation state. For the plane-strain problem, the components of the deformation gradient tensor of a simple-shear deformation state are

$$F_{11} = F_{22} = 1, \quad F_{12} = 0, \quad F_{21} = \tan\gamma, \tag{4}$$

where γ is the shear angle. Our theoretical analysis shows that the S-wave path will deform in accordance with the material coordinate curve, while the P-wave will follow a different path, rendering a separation of the two waves. As a result, the entangled elastic waves are imported from one lateral side of the neo-Hookean solid and the separated pure waves can be received on the other side, as illustrated in Fig. 2(a).

To validate the effectiveness of the proposed wave-mode splitting method, numerical experiments are performed by using the software COMSOL Multiphysics.[18] In the first example, we consider a nearly incompressible neo-Hookean material (PSM-4 (Ref. 21)) of cuboid shape with

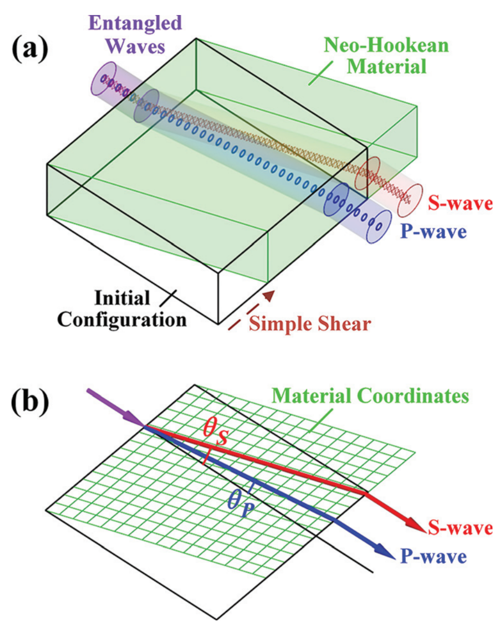

FIG. 2. (a) Principle of the wave-mode splitting method. The neo-Hookean material undergoes homogeneous simple-shear deformation. The mesh in (b) represents the deformed material coordinate curves. The red and blue arrows denote the propagating paths of S- and P-waves, respectively.

side length $l_s = 0.12\,\text{m}$. Take the material parameters $\lambda = 2\,\text{GPa}$, $\mu = 1.08\,\text{MPa}$, and $\rho = 1050\,\text{Kg/m}^3$. A homogeneous simple-shear deformation is specified by applying the displacement field $U_y = (x + 0.6)/30\,\text{m}$ in the body. The corresponding deformation gradient is defined in Eq. (4) with $\tan\gamma = 1/3$, and the deformed material coordinate curves are plotted in Fig. 2(b). The homogeneous von Mises stress in the material is $6.35 \times 10^5\,\text{N/m}^2$. On the left boundary of the device, an in-plane harmonic Gaussian S-wave beam and a P-wave beam are imported at the same position along the horizontal direction. Both the incident waves are set the amplitude $u = a\exp(-r^2/w^2)\text{m}$, with $a = 1 \times 10^{-3}\,\text{m}$, $r = 0.2\sqrt{\pi}(y - 0.2)\text{m}$, and $w = 0.025\,\text{m}$. The angular frequencies of the S-wave and the P-wave are $\omega = 0.3\,\text{MHz}$ and $\omega = 12.9\,\text{MHz}$, respectively.

To characterize the two wave modes, we calculate the divergence $S_D = u_{x,x} + u_{y,y}$ and the curl $S_C = u_{x,y} - u_{y,x}$ of the displacement field in the device. Let the normalized parameters $N_P = \text{abs}(S_D)/\max(S_D)$ and $N_S = \text{abs}(S_C)/\max(S_C)$ to describe the spatial intensities of P- and S-wave modes, respectively. As illustrated in Fig. 3(a), the in-plane S-wave propagates along with the deformed mesh in the neo-Hookean material, while the deformation has essentially ignorable influence on the propagation of the P-wave. The distributions of the normalized wave parameters N_P and N_S on the right boundary of the computational domain are shown in Fig. 3(b). For comparison, the wave fields of the aforementioned elastic wave beams propagating through an undeformed neo-Hookean material are calculated and the corresponding N_P and N_S are illustrated in Fig. 3(c). It is seen that the magnitudes of N_P and N_S are almost the same between the cases of simple-sheared [Fig. 3(b)] and undeformed [Fig. 3(c)] neo-Hookean materials. This implies that the proposed wave-mode splitting device is almost energy-lossless. Also importantly, this method is broadband because no dispersive mechanism is involved in the homogeneous neo-Hookean material.

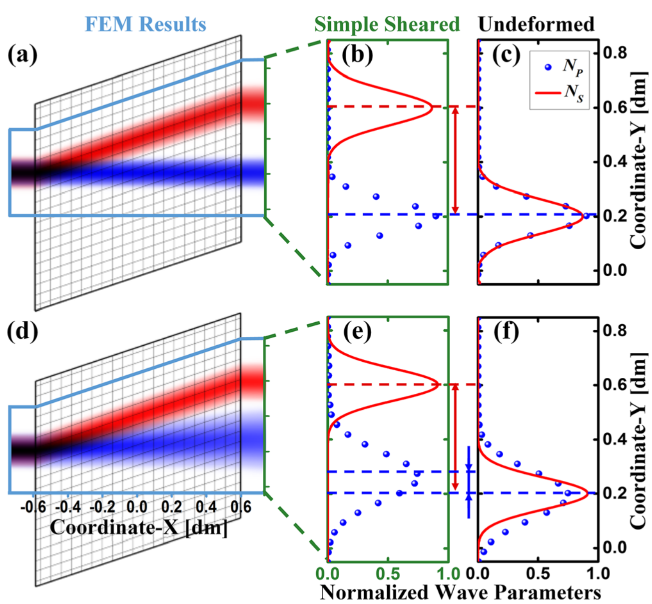

FIG. 3. (a) Normalized wave parameters N_P and N_S in a nearly incompressible neo-Hookean material under simple shear. (b) The distributions of N_P and N_S on the right boundary of the computational domain. For comparison, N_P and N_S of the elastic waves propagating through an undeformed neo-Hookean material are given in (c). Figures (d), (e), and (f) are the corresponding results in a compressible neo-Hookean material.

It is worth noticing that the P-wave path in a deformed neo-Hookean material is not always along the original straight line but may have a slight derivation, which depends on the material parameters λ and μ. In the case when both the material and its deformation are homogeneous, the P-wave beam will propagate along a straight line with a small inclined angle θ_P measured from the original direction, as shown in Fig. 2(b). In this sense, the direction of the P-wave beam will be determined only by the refraction angle on the interface between the free space and the simple-sheared neo-Hookean material.

According to Ogden[16] and Auld,[22] the speeds of elastic waves propagating in a uniformly deformed neo-Hookean solid are (for their derivations, see the supplementary material[18])

$$c_P = \sqrt{\frac{\tilde{\lambda} + \tilde{\mu}_1 + \tilde{\mu}_2}{\rho_0}}, \quad c_S = \sqrt{\frac{\tilde{\mu}_2}{\rho_0}}, \tag{5}$$

where $\rho_0 = J^{-1}\rho$, $\tilde{\lambda} = \lambda(2J - 1)$, $\tilde{\mu}_1 = \lambda(1 - J) + J^{-1}\mu$, and $\tilde{\mu}_2 = J^{-1}\mu F_{i's}F_{k's}l_{i'}l_{k'}$, with ρ being the initial mass density of the material and l_i being the unit vector in the wave propagation direction. The wave paths refracted from an interface between an undeformed and a deformed hyperelastic material can be determined from the phase-slowness curves.[22,23] For a homogeneously simple-sheared neo-Hookean solid, the horizontally incident elastic waves on its left boundary have a refraction angle of $\theta_P = \arctan(\eta\tan\gamma)$, where $\eta = \mu/(\lambda + 2\mu)$ and $\theta_S = \gamma$.[18] For soft materials that can undergo large deformations, the parameter η is usually in the range from 0 to 0.4.[24] For the example of $\tan\gamma = 1/3$, the refraction angles of P- and S-waves are plotted as functions of η in Fig. 4. It is seen that θ_S is always 18.4°, in accordance with the result from the transformation theory. For the P-wave, the refraction angle is approximately proportional to η. For a nearly incompressible neo-Hookean material ($\eta \approx 0$), the refraction angle approximates to zero, in consistency with our numerical simulation [Fig. 3(a)]. When the hyperelastic material is compressible, the refraction

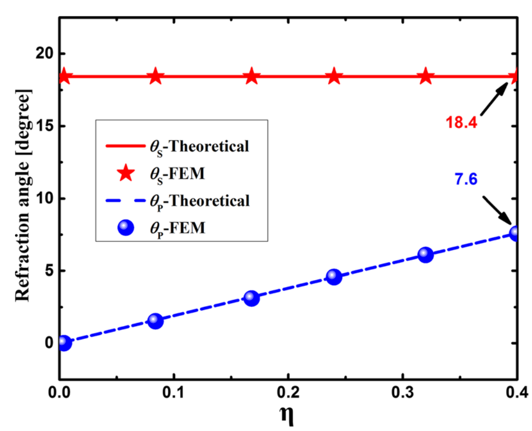

FIG. 4. Refraction angles θ_P and θ_S of P- and S-waves in the case of $\tan\gamma = 1/3$. The lines represent the theoretical solution and the scatters are the numerical simulation results. The material parameter η is defined by $\eta = \mu/(\lambda + 2\mu)$.

angle θ_P increases with η but is always smaller than θ_S. For example, Figs. 3(d)–3(f) give the simulation results for the propagation of the S-wave and the P-wave in a compressible neo-Hookean material, where we take the material parameters $\lambda = 4.32\,\text{MPa}$, $\mu = 1.08\,\text{MPa}$, and $\rho = 1050\,\text{Kg/m}^3$, which correspond to $\eta = 1/6$. The angular frequencies of the S-wave and the P-wave are both $\omega = 0.3\,\text{MHz}$. The amplitudes of the incident waves are the same as those in Figs. 3(a)–3(c). Comparing the distributions of N_P in the simple-sheared material [Fig. 3(e)] and the undeformed material [Fig. 3(f)], a slight P-wave shift of $6.32 \times 10^{-3}\,\text{m}$, which corresponds to $\theta_P = 3.1^\circ$, can be observed. The calculation results for the refraction angles of P- and S-waves are plotted in Fig. 4, showing a good agreement with our theoretical prediction. The above analysis demonstrates the feasibility of the wave-mode splitting method. In addition, a transient analysis of the above problem is also performed[18] and the results are consistent with the above theoretical prediction and the steady state analysis.

Furthermore, it is emphasized that pre-deformation is not necessary to be homogeneous in the proposed wave-mode separation device. According to the hyperelastic transformation theory, an S-wave propagating in a neo-Hookean material with inhomogeneous pre-deformation always attaches the deformed material coordinate curves, while the path of P-wave is slightly affected by the inhomogeneity of deformation. However, by properly choosing the location of the wave inlet, this effect can be negligible. To verify this, another variant of the wave-mode splitting device is also modeled (supplementary material[18]). In this model, the left boundary is fixed and a uniform displacement of $U_y = 0.04\,\text{m}$ is applied on the right boundary. Our simulations show that for both compressible and incompressible materials, the inhomogeneity of shear deformation does not affect the wave-splitting function of the device.

In summary, we have demonstrated that a pre-deformed neo-Hookean material can behave like a smart metamaterial and can be utilized to manipulate both in-plane and anti-plane S-waves. This enables us to design tunable and broadband soft wave-transformation devices. Owing to the different behaviors of P- and S-waves in deformed neo-Hookean solids, a simple and convenient wave-mode splitting device is proposed in this letter by using a soft hyperelastic material under simple-shear deformation. Such a tunable device is broadband, energy-lossless, and thus holds promise for significant applications in such fields as seismic protection, structural health monitoring, and biomedical imaging. Finally, it is emphasized that the viscoelastic effect of the material, which has been neglected in this study, may affect the performance of the proposed soft device and a low-damping hyperelastic material should be used in practical applications.

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