Transformation ray method: Controlling high frequency elastic waves (L)

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(Received 28 October 2011; revised 8 April 2012; accepted 19 June 2012)

Elastic ray theory is a high frequency asymptotic approximation of solution of elastodynamic equation, and is widely used in seismology. In this paper, the form invariance under a general spatial mapping and high frequency wave control have been examined by transformation method. It is showed that with the constraint of major and minor symmetry of the transformed elastic tensor, the eikonal equation keeps its form under a general mapping, however, the transport equation loses its form except for conformal mapping. Therefore, the elastic ray path can be controlled in an exact manner by a transformation method, whereas energy distribution along the ray is only approximately controlled. An elastic rotator based on ray tracing method is also provided to illustrate the method and to access the approximation.
 [http://dx.doi.org/10.1121/1.4744973]

PACS number(s): 43.40.Sk, 43.20.Dk, 43.35.Cg, 43.40.Ph [ANN] Pages: 2942–2945

I. INTRODUCTION

Transformation methods proposed by Greenleaf *et al.*,[1] Pendry *et al.*,[2] and Leonhardt,[3] pave the way for finding material spatial distribution when the function of an electromagnetic device is defined by spatial mapping. The same idea is also applied to an acoustic wave as the Helmholtz equation is form invariant.[4–9] However, for elastic waves in solid materials, Milton *et al.* show that elastodynamic equation (Navier's equation) is transformed to Willis' equation under a general spatial mapping,[10] therefore, the techniques developed for electromagnetic or acoustic waves cannot be applied exactly to control elastic waves. Several works have been conducted to control elastic waves by distributing spatially materials.[11–14]

More recently, Hu *et al.*[15] proposed an approximate method to establish the transformation relations for elastic waves, their method is based on local invariance of elastodynamic equation (Navier's equation) and energy conservation condition by idealizing a general spatial mapping with a series of local affine ones. It is found that the predicted results are better for high frequency elastic waves. High frequency elastic waves in heterogeneous solids can be analyzed within elastic ray theory,[16] which is governed by eikonal and transport equations, this motivates us to examine the transformation method in the context of elastic ray theory. It is also hoped that this work can clarify the nature of the approximation made in Ref. 15. This letter starts by introducing briefly the elastic ray theory, then the form invariance of the governing equation is examined, and the ray tracing method in the case of anisotropic mass is also explained. Finally, the method is illustrated by a numerical example.

II. BACKGROUND ON ELASTIC RAY THOERY

We start from the general elastodynamic equation (Navier's equation) with a mass of tensor form[17]

$$(C_{ijkl}u_{k,l})_{,j} = -\rho_{ij}\ddot{u}_j, \tag{1}$$

where $\mathbf{u}$ is the displacement vector and $\mathbf{C}$ is the rank-four elastic tensor. The mass density ρ is a second-order tensor in order to include possible metamaterials. In a smoothly inhomogeneous media, a high-frequency elastic wave can be approximately separated into quasi-compression (qP) wave and quasi-shear (qS1 and qS2) waves. In this case, to solve Eq. (1), a time-harmonic solution is represented in form of ray series,[16]

$$u_i(x_j, t) = \left[\sum_{n=0}^{\infty}\frac{U_i^{(n)}(x_j)}{(-i\omega)^n}\right]\exp(-i\omega(t - T(x_j))), \tag{2}$$

where $\mathbf{U}^{(n)}$ is the amplitude vector of the nth order and T is a scalar function, called travel time (eikonal). In the following discussion, we will follow the zeroth-order approximation of ray series.[16]

A harmonic plane wave solution of the displacement vector is assumed to be of the following form:

$$u_i(x_j, t) = U_i(x_j)\exp(-i\omega(t - T(x_j))). \tag{3}$$

Inserting Eq. (3) into Eq. (1) yields

$$(C_{ijkl}U_{k,l})_{,j} + i\omega[C_{ijkl}U_{k,l}T_{,j} + (C_{ijkl}U_kT_{,l})_{,j}] - \omega^2[(C_{ijkl}U_kT_{,l}T_{,j}) - \rho_{ij}U_j] = 0. \tag{4}$$

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Equation (4) should be satisfied for any frequency, so the coefficients with ω^n ($n = 0,\ 1,\ 2$) must vanish, we get

$$(C_{ijkl}U_kT_{,l}T_{,j}) - \rho_{ij}U_j = 0, \tag{5a}$$

$$C_{ijkl}U_{k,l}T_{,j} + (C_{ijkl}U_kT_{,l})_{,j} = 0, \tag{5b}$$

$$(C_{ijkl}U_{k,l})_{,j} = 0. \tag{5c}$$

For high frequency elastic waves ($\omega \gg \omega^0 = 1$), the first term in Eq. (4) (with ω^0) can be neglected compared to the second (with ω^1) and third (with ω^2) terms, therefore, it can be safely dropped. So, the governing equations of elastic ray theory consist of Eqs. (5a) and (5b); Eq. (5a) is the governing equation for the path of elastic ray, called eikonal equation; whereas Eq. (5b) monitors the energy transfer along the ray, called transport equation. Although the transformation method based on Eq. (1) has been discussed in different works,[10,11,15,18] the transformation method based on Eqs. (5a) and (5b) has not been examined so far.

III. THEORETICAL FORMULATION

A. Transformation method for elastic ray theory

Consider a general mapping $x'_{i'} = x'_{i'}(x_j)$, Eq. (5a) is transformed into x' coordinates with a change of variable as

$$(\beta_j^{j'}\beta_l^{l'}C_{ijkl}U_kT_{,l'}T_{,j'}) - \rho_{ij}U_j = 0, \tag{6}$$

where $\beta_i^{i'} = \partial x'_{i'}/\partial x_i$ is the component of Jacobian matrix of the coordinate transformation, T is a scalar independent on the coordinates. Similarly, expressing Eq. (5b) in x' coordinates, and introducing the Jacobian J of the coordinate transformation, and with help of the relation $(J^{-1}\beta_{i'}^{i})_{,i} = 0$,[6,19] we get

$$J^{-1}\beta_j^{j'}\beta_l^{l'}C_{ijkl}U_{k,l'}T_{,j'} + (J^{-1}\beta_j^{j'}\beta_l^{l'}C_{ijkl}U_kT_{,l'})_{,j'} = 0. \tag{7}$$

Compared to Eqs. (5a) and (5b), form invariance of Eqs. (6) and (7) leads to the following transformation relations:

$$C'_{i'j'k'l'} = J^{-1}\beta_j^{j'}\beta_l^{l'}C_{ijkl}, \tag{8a}$$

$$\rho'_{i'j'} = J^{-1}\rho_{ij}, \tag{8b}$$

$$U'_{i'} = U_i. \tag{8c}$$

It is interesting to note that these transformation relations are identical with some former works,[11,18] derived from the elastodynamic equation, however, the elastic tensor loses its minor symmetry.

So in the following, we seek the transformation relations with symmetric elastic tensor. Based on Eq. (8a), the symmetric elastic tensor in the transformed space should have the following form:

$$C'_{i'j'k'l'} = J^{-1}\beta_i^{i'}\beta_j^{j'}\beta_k^{k'}\beta_l^{l'}C_{ijkl}. \tag{9a}$$

With Eq. (9a) and the form invariance of Eq. (5a) under a general mapping, the transformation relations of the mass density and displacement now become accordingly as

$$\rho'_{i'j'} = J^{-1}\beta_i^{i'}\beta_j^{j'}\rho_{ij}, \tag{9b}$$

$$U'_{i'} = \beta_{i'}^{i}U_i. \tag{9c}$$

Transformation relation (9) makes Eq. (5a) form invariant, unfortunately not for Eq. (5b). Therefore the transformed material parameters given by Eq. (9) can be used to control exactly the elastic ray path defined by the mapping, and only approximately to control the energy distribution along the ray. Inserting Eq. (9) into Eq. (5b), after some algebra, we obtain

$$C_{ijkl}(U_k)_{,l}T_{,j} + (C_{ijkl}U_kT_{,l})_{,j} + \beta_{i'}^{m}(\beta_i^{i'})_{,j}C_{ijkl}U_kT_{,l} = 0. \tag{10}$$

Compared to Eq. (5b), an additional term related to the mapping and its derivative appears. Obviously, this extra term vanishes when the mapping approaches to bilinear ($(\beta_i^{i'})_{,j} = 0$). So if we use Eq. (9) to approximately control energy distribution along ray by the mapping technique, the error will be smaller for smoother mapping and higher frequency. We note that Eq. (9) is the same as those obtained in Ref. 15 by a complete different method, therefore this implies that the approximation made in Ref. 15 lies in high frequency assumption through local affine transformation. As the approximate high-frequency solution of the elastodynamic equation, the elastic ray theory requires that the appropriate material parameters of the medium vary smoothly over a distance of the order of wavelength λ.[16]

Here we also note that in a 2-D case, when the spatial mapping is conformal, the extra term in Eq. (10) can be eliminated ($\beta_{i'}^{m}(\beta_i^{i'})_{,j} = 0$) due to Cauchy–Riemann conditions of a conformal mapping. In this case, both ray path and energy distribution along the ray can be controlled exactly with Eq. (9).

B. Ray tracing method for a transformation medium

To illustrate the theory, the elastic ray path and energy distribution have to be evaluated for a device defined by spatial mapping. Due to the anisotropic mass density, classical elastic ray tracing method must be reexamined. Equation (5a) can be expressed as

$$\Gamma_{sk}U_k = 0, \tag{11}$$

where $\Gamma_{sk} = \rho_{is}^{-1}C_{ijkl}p_lp_j - \delta_{sk}$ is called Christoffel matrix, whereas ρ_{is}^{-1} and $p_i = T_{,i}$ are the inverse of the mass density matrix and the component of slowness vector, respectively. Compared with the classical ray tracing method, the only difference consists in Γ_{sk}, so we can follow the same procedures to calculate the ray path. Based on Eq. (11), the path of an incident ray in a smoothly inhomogeneous anisotropic media can be calculated from the eigenvalues G_i and eigenvectors $\vec{g}_i$ of the Christoffel matrix. After determining the initial position and slowness vector of an incident ray, the ray path can be obtained by the following relations:

$$dx_i/dT = (1/2)(\partial G_m/\partial p_i), \tag{12a}$$

$$dp_i/dT = -(1/2)(\partial G_m/\partial x_i), \tag{12b}$$

where $m = 1,\ 2,\ 3$ refer qS1 qS2, and qP waves, respectively. The travel time T is chosen to be the parameter along the ray.

To evaluate the energy transfer along the ray, Eq. (5b) is multiplied by g_i and rearranged into

$$2V_iA_{,i} + A(V_i)_{,i} = 0, \tag{13}$$

where A is the scalar, complex-valued amplitude function, which relates displacement to eigenvector by $U_i = Ag_i$, and

$$V_i = C_{ijkl}p_l g_k g_j. \tag{14}$$

Equation (13) is called transport equation for an inhomogeneous medium. In a classical ray tracing method,[16] Eq. (13) is evaluated by introducing an elementary ray tube with a specified ray and a family of rays nearby, as shown in Fig. 1. As V_i has the same direction as the unit ray vector t_i,[16] divergence theorem can then be applied on $(V_i)_{,i}$ for ease of calculation. However, for a transformation medium, the coincidence of the directions of $V'_{i'}$ and $t'_{i'}$ should be reexamined due to the introduced anisotropic mass. To this end, we analyzed the directions of these two vectors with the help of the coordinate transformation. Under a coordinate transformation, a vector $a_i(x_i)$ is transformed into $a'_{i'}(x'_{i'}) = \beta_i^{i'} a_i(x'_{i'})$. Similarly, a ray vector $t_i(x_i)$ in a virtual space will be transformed into $t'_{i'}(x'_{i'}) = \beta_i^{i'} t_i(x'_{i'})$ in a physical space. On the other hand, we have

$$\begin{aligned} V'_{i'} &= J^{-1}\beta_i^{i'}\beta_j^{j'}\beta_k^{k'}\beta_l^{l'}C_{ijkl}\beta_{l'}^{r}T_{,r}\beta_{k'}^{s}g_s\beta_{j'}^{t}g_t \\ &= J^{-1}\beta_i^{i'}C_{ijkl}p_l g_k g_j = J^{-1}\beta_i^{i'}V_i\,. \end{aligned} \tag{15}$$

As in the virtual space, the vector V_i has the same direction as t_i,[16] therefore $V'_{i'}$ in the physical space will follow the same direction as $t'_{i'}$, as well after the transformation. In this context, after expressing Eq. (11) in ray coordinates, we can apply a divergence theorem on $(V_i)_{,i}$ in the elementary ray tube, therefore, the same process as classical elastic ray method can then be followed. The amplitudes along the ray can be obtained by using[16]

$$A(T_{n+1}) = \left[\frac{\Omega^{\perp}(T_n)}{\Omega^{\perp}(T_{n+1})}\right]^{1/2} A(T_n), \tag{16}$$

where $\Omega^{\perp}$ is the cross-sectional area of the elementary ray tube in a 3-D case, as shown in Fig. 1, whereas in a 2-D case, $\Omega^{\perp}$ is degenerated into the length of one line element connecting two points situated on two close rays.

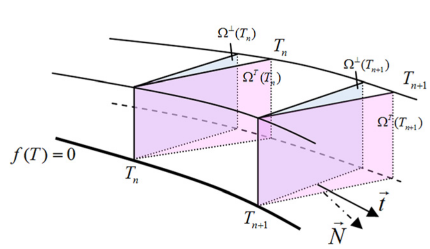

FIG. 1. (Color online) A sketch of an elementary ray tube in inhomogeneous anisotropic medium with an elastic ray $f(T) = 0$ and a family of rays nearby. T_n and T_{n+1} are two points on the ray used in the recurrent computation. Ω^T denotes a cut from the ray tube by the wave fronts and $\Omega^{\perp}$ is the cross section of the ray, which can be calculated through some geometry relation by using $\Omega^T.\vec{t}$ is the unit vector perpendicular to $\Omega^{\perp}$, representing the direction of the ray.

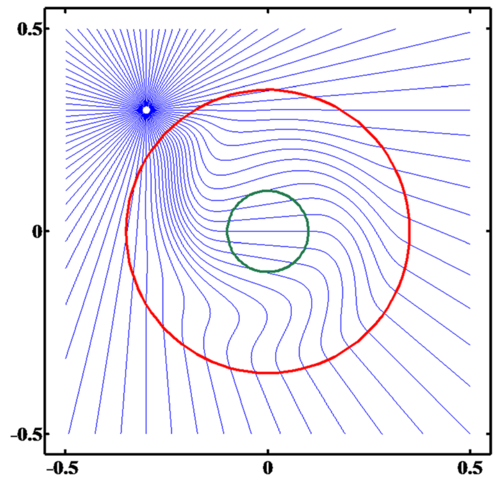

FIG. 2. (Color online) Ray traces of the elastic rotator calculated by transformation ray method, they are same as those evaluated by FEM simulation.

IV. NUMERICAL EXAMPLES

In the following, a 2-D elastic rotator is examined by the ray tracing method, the second one is realized by a general spatial mapping. The elastic ray path and energy along the ray will be evaluated with the functions defined by the spatial mappings.

Elastic rotator can guide elastic wave to rotate for a certain angle. The design can be taken via transformation technique by using a general mapping. The detailed parameters of the device are the same as those in Ref. 15 for a comparison purpose. Using Eqs. (12) and (16), the ray paths and amplitudes in the rotator are calculated by MATLAB software. 80 rays of the S-wave with a wave length $l = 0.015\,\mathrm{m}$ emitted from the source are traced, the results are shown in Fig. 2. A FEM simulation based on the elastodynamic equation is also

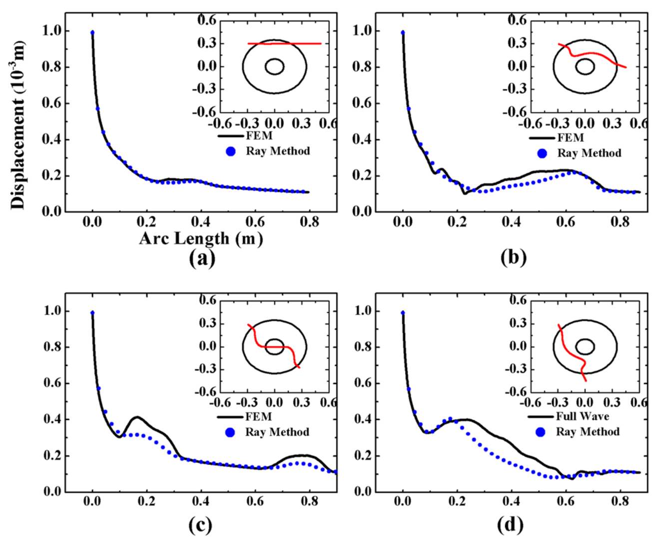

FIG. 3. (Color online) Total displacement along the rays evaluated by transformation ray theory (symbols), FEM simulation (solid line) at incident angles: (a) 0°, (b) −22.5°, (c), −45°, and (d) −67.5°.

performed, the calculated ray path (not shown in Fig. 2) is shown to be identical as that predicted by our transformation ray theory, as expected. The total displacements $|\sqrt{U_1^2 + U_2^2}|$ (representing energy) on the four specific rays with incidence angles 0°, -22.5°, -45°, and -67.5° calculated by using transformation ray method (16) (labeled by ray method) and FEM simulation (labeled by FEM) are given in Fig. 3. The results show that although some errors occur in the rotator, overall the amplitudes calculated by the ray tracing method agree reasonably well with the FEM simulation. The same results can also be found for P waves. These examples confirm our theoretical founding: Transformation relation (9) can be used to control exactly the ray path, also approximately well to control the energy distribution along the ray.

V. CONCLUSIONS

In summary, we have examined the transformation method in the context of the elastic ray theory for high frequency elastic waves. It is found that the eikonal equation is form invariant under a general mapping with a symmetric transformed elastic tensor, however, the transport equation is not form invariant, except for a conformal mapping. Therefore, the ray path can be controlled exactly with the transformation technique, but not for the amplitude along the ray. An elastic rotator is examined with the ray tracing method extended to the case of anisotropic mass density, the numerical results confirm the theoretical founding. The obtained transformation relations are identical to those in Ref. 15 by a completely different method, this coincidence clarifies the nature of local affine approximation made in Ref. 15. Application of the proposed method can be anticipated to seismic protection and health monitoring technique where the high frequency approximation remains valid.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (10832002, 11172037, 10972036) and the National Basic Research Program of China (2011CB610302) and Excellent Young Scholars Research Fund of Beijing Institute of Technology.

[1] A. Greenleaf, M. Lassas, and G. Uhlmann, "On non-uniqueness for Calderón's inverse problem," Math. Res. Lett. **10**, 685–693 (2003).
[2] J. B. Pendry, D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," Science **312**, 1780–1782 (2006).
[3] U. Leonhardt, "Optical conformal mapping," Science **312**, 1777–1780 (2006).
[4] H. Chen and C. T. Chan, "Acoustic cloaking in three dimensions using acoustic metamaterials," Appl. Phys. Lett. **91**, 183518 (2007).
[5] S. A. Cummer and D. Schurig, "One path to acoustic cloaking," New J. Phys. **9**, 45 (2007).
[6] A. N. Norris, "Acoustic cloaking theory," Proc. R. Soc. London, Ser. A **464**, 2411–2434 (2008).
[7] A. N. Norris, "Acoustic metafluids," J. Acoust. Soc. Am. **125**, 839–849 (2009).
[8] C. Ren, Z. Xiang, and Z. Cen, "Design of acoustic devices with isotropic material via conformal transformation," Appl. Phys. Lett. **97**, 044101 (2010).
[9] S. Zhang, C. Xia, and N. Fang, "Broadband acoustic cloak for ultrasound waves," Phys. Rev. Lett. **106**, 024301 (2011).
[10] G. W. Milton, M. Briane, and J. R. Willis, "On cloaking for elasticity and physical equations with a transformation invariant form," New J. Phys. **8**, 248 (2006).
[11] M. Brun, S. Guenneau, and A. B. Movchan, "Achieving control of in-plane elastic waves," Appl. Phys. Lett. **94**, 061903 (2009).
[12] X. M. Zhou, G. K. Hu, and T. J. Lu, "Elastic wave transparency of a solid sphere coated with metamaterials," Phys. Rev. B **77**, 024101 (2008).
[13] M. D. Guild, A. Alu, and M. R. Haberman, "Cancellation of acoustic scattering from an elastic sphere," J. Acoust. Soc. Am. **29**, 1355–1365 (2011).
[14] A. V. Amirkhizi, A. Tehranian, and S. Nemat-Nasser, "Stress-wave energy management through material anisotropy," Wave Motion **47** 519–536 (2010).
[15] J. Hu, Z. Chang, and G. K. Hu, "Approximate method for controlling solid elastic waves by transformation media," Phys. Rev. B **84**, 201101(R) (2011).
[16] V. Cevery, *Seismic Ray Theory* (Cambridge University Press, New York, 2001), Chap. 2, pp. 62–63 and Chap. 3, pp. 138–148.
[17] Z. Liu, C. T. Chan, and P. Sheng, "Analytical model of phononic crystals with local resonances," Phys. Rev. B **71**, 014103 (2005).
[18] A. Norris and A. L. Shuvalov, "Elastic cloaking theory," Wave Motion **48**, 525–538 (2011).
[19] R. W. Ogden, *Non-Linear Elastic Deformations* (Dover, New York, 1997), Chap. 2, pp. 86–89.