Acta Mechanica Solida Sinica, Vol. 27, No. 5, October, 2014 ISSN 0894-9166
Published by AMSS Press, Wuhan, China

HEAT FLOW CONTROL BY TRANSFORMATION METHOD WITH GRID GENERATION METHOD**

Zheng Chang[1,2] **Xiaoning Liu**[1*] **Gengkai Hu**[1*]

([1] *Key Laboratory of Dynamics and Control of Flight Vehicle, Ministry of Education, School of Aerospace Engineering, Beijing Institute of Technology, Beijing* 100081, *China*)

([2] *CNMM & AML, Department of Engineering Mechanics, Tsinghua University, Beijing* 100084, *China*)

Received 27 December 2012, revision received 21 June 2013

ABSTRACT In the context of the transformation method, we propose a general approach to construct numerically the mapping generated by imposing specific boundary conditions with a targeted function, and the necessary material and heat source spatial distributions are then derived with the help of transformation method. The construction of mapping by grid generation method through solving partial differential equations circumvents the limitation of device geometry, which paves the way for designing more complex heat flow control devices. Two numerical examples are also given to show how to design material properties and heat source in order to control temperature patterns.

KEY WORDS heat conduction, inverse problem, transformation method, grid generation

I. INTRODUCTION

Heat flow control is an important issue for engineering applications, including thermal shielding in hypersonic vehicles, thermal management for electronic systems and many others. The problem consists of finding material spatial distributions in order to produce a desired heat flow or temperature pattern with a targeted application, and this usually is classified as an inverse heat transfer problem (IHTP). Many numerical, analytical, and semi-analytical approaches have been developed for solving IHTPs[1–5]. Explicit analytical solutions are only limited to simple geometries. For more complex cases, an optimization procedure is usually taken which is time consuming.

In recent years, the transformation method provides a novel technique for finding in a direct way the necessary material distributions if a function is prescribed[6], and this method applies to any physical system with form invariance of spatial transformation. Proposed initially for electrodynamics by Pendry[7] and Leonhardt[8], this method has quickly found applications in control of acoustic/elastic waves[9–12], matter waves[13] by distributing materials in space. For a heat conduction problem, the governing equation has the same form as that of electrostatics, which is shown to be form invariant[14]. Therefore, it is natural to apply this method to design material distributions for heat flow control. In fact, based on this method, Fan et al.[15] constructed an ellipsoidal cloak to control heat flow to go around a cloaked region, Guenneau et al.[16] provided a detailed analysis on transformation thermodynamics, and some interesting thermal devices are also proposed. Experimental works are

* Corresponding author. E-mail: liuxn@bit.edu.cn; hugeng@bit.edu.cn

** Project supported by the National Natural Science Foundation of China (Nos. 11372035, 10832002 and 11221202) and China Postdoctoral Science Foundation (No. 2014M550054).

recently conducted to validate the concept of heat flow or temperature pattern control based on the transformation method[17,18].

Until now the methods mentioned above for controlling heat flow are limited to simple geometries where analytical transformation functions are available. Therefore, it is necessary to develop a general method based on the transformation method to manage heat flow with complex geometries, and this is the objective of our work. As well known for electrodynamics, the spatial mapping of a complex transformation device can be obtained by solving numerically PDEs, e.g. Laplace equation[19]. By adjusting the boundary condition, design with quasi-isotropic material can also be achieved[20]. In this paper, we will utilize this numerical method to evaluate the transformation function, and together with the transformation method we will propose a general method to manipulate heat flow. The paper is arranged as follows: In §II, the form invariance of the transient heat conduction equation will be examined and the transformation relation for controlling heat flow will be provided. In addition, the construction of transformation by a grid generation method will also be explained. Numerical examples will be presented in §III to illustrate the proposed method. Some concluding remarks will be made in §IV.

II. THEORETIAL ANALYSIS OF TRANSFORMATION-BASED HEAT FLOW CONTROL WITH GRID GENERATION METHOD

2.1. Transformation-based Heat Flow Control

Consider the following transient heat conduction equation:

$$C(\boldsymbol{x})\frac{\partial T(\boldsymbol{x},t)}{\partial t}+\nabla\cdot\boldsymbol{q}(\boldsymbol{x},t)=s(\boldsymbol{x},t)\quad \text{on}\quad \Omega \tag{1}$$

where T is the temperature, C the heat capacity, and s the heat source. $\boldsymbol{q}$ is the heat flux which is related to temperature gradient by Fourier's law

$$\boldsymbol{q}(\boldsymbol{x},t)=-\boldsymbol{k}(\boldsymbol{x})\cdot\nabla T(\boldsymbol{x},t) \tag{2}$$

where $\boldsymbol{k}$ is the heat conductivity tensor. To establish a well-posed problem, the following Dirichlet and Newmann boundary conditions on $\partial\Omega$, and the initial condition on Ω must be specified

$$T(\boldsymbol{x},t)=T_B(\boldsymbol{x},t)\quad \text{on}\quad \partial\Omega^T \tag{3a}$$

$$\boldsymbol{q}(\boldsymbol{x},t)\cdot\boldsymbol{n}=q_B(\boldsymbol{x},t)\quad \text{on}\quad \partial\Omega^q \tag{3b}$$

$$T(\boldsymbol{x},0)=T_0(\boldsymbol{x})\quad \text{in}\quad \Omega \tag{3c}$$

where $\boldsymbol{n}$ is the outward unit normal vector of the boundary, T_B and q_B the temperature and heat flux on the boundaries, respectively. T_0 is the initial temperature in the domain Ω.

Following the transformation method, we consider a spatial mapping $\boldsymbol{x}'=\boldsymbol{x}'(\boldsymbol{x})$, which maps the original flat Cartesian space $\boldsymbol{x}$ to a curved space (or transformed space) $\boldsymbol{x}'$. It has been proved that Eqs.(1) and (2) are form invariant under a general transformation[14]. Therefore if the space was spanned with any curved coordinate, the governing equations take just the same form as those in the Cartesian frame

$$C'(\boldsymbol{x}')\frac{\partial T'(\boldsymbol{x}',t)}{\partial t}+\nabla'\cdot\boldsymbol{q}'(\boldsymbol{x}',t)=s'(\boldsymbol{x}',t) \tag{4}$$

$$\boldsymbol{q}'(\boldsymbol{x}',t)=-\boldsymbol{k}'(\boldsymbol{x}')\cdot\nabla'T'(\boldsymbol{x}',t) \tag{5}$$

where $\nabla'=\left[\partial/\partial x_1{}'\ \ \partial/\partial x_2{}'\ \ \partial/\partial x_3{}'\right]^{\mathrm{T}}$. In this transformed space, the physical quantities and material constants are simultaneously transformed from the initial flat Cartesian space. By examining the form invariance of Eqs.(4) and (5), the temperature T and heat flux $\boldsymbol{q}$ in the original space are found to be mapped to

$$T'(\boldsymbol{x}')=T[\boldsymbol{x}(\boldsymbol{x}')],\quad \boldsymbol{q}'(\boldsymbol{x}')=\frac{\boldsymbol{A}\cdot\boldsymbol{q}(\boldsymbol{x})}{\det(\boldsymbol{A})} \tag{6}$$

where $\boldsymbol{A}$ is the Jacobian matrix

$$A_{ij}=\frac{\partial x_j'}{\partial x_i} \tag{7}$$

At the same time, the conductivity, heat capacity and source term are mapped to the transformed space as

$$\boldsymbol{k}'(\boldsymbol{x}') = \frac{\boldsymbol{A}\cdot\boldsymbol{k}(\boldsymbol{x})\cdot\boldsymbol{A}^{\mathrm{T}}}{\det(\boldsymbol{A})}, \quad C'(\boldsymbol{x}') = \frac{C(\boldsymbol{x})}{\det(\boldsymbol{A})}, \quad s'(\boldsymbol{x}') = \frac{s(\boldsymbol{x})}{\det(\boldsymbol{A})} \tag{8}$$

Different from the transformation optics or acoustics, where the interested transformed region is usually embedded in an infinite open domain, in the transformation heat conduction, the transformation of boundary and initial conditions has to be taken into account. Substituting Eq.(6) and transformed boundary unit normal $\boldsymbol{n}' = \boldsymbol{A}^{-\mathrm{T}}\cdot\boldsymbol{n}$ into Eq.(3), we obtain the invariant form of the boundary and initial conditions as the following:

$$T'(\boldsymbol{x}') = T'_B(\boldsymbol{x}') \quad \text{on} \quad \partial\Omega'^T \tag{9a}$$

$$\boldsymbol{q}'(\boldsymbol{x}')\cdot\boldsymbol{n}' = q'_B(\boldsymbol{x}') \quad \text{on} \quad \partial\Omega'^q \tag{9b}$$

$$T'(\boldsymbol{x}',0) = T'_0(\boldsymbol{x}') \quad \text{on} \quad \Omega' \tag{9c}$$

where

$$T'_B(\boldsymbol{x}') = T_B[\boldsymbol{x}(\boldsymbol{x}')], \quad q'_B(\boldsymbol{x}') = \frac{q_B(\boldsymbol{x})}{\det(\boldsymbol{A})}, \quad T'_0(\boldsymbol{x}',0) = T_0[\boldsymbol{x}(\boldsymbol{x}')]$$

Due to the form invariance of the governing equations and boundary/initial conditions, the action of the spatial transformation may have two different interpretations: One is just description of the same physical process in the curved space described by Eqs.(4), (5) and (9), which is shown in Fig.1(a). The other is to interpret the coordinate $\boldsymbol{x}'$ in the curved space as if in a Cartesian frame, and the effect of the spatial mapping is now mimicked by new material and boundary/initial conditions defined by Eqs.(8) and (9), which is called geometry-material equivalency[7]. The second interpretation also endows the spatial transformation a deformation meaning (Fig.1(b)), and a new heat flow pattern (a deformed heat flow $\boldsymbol{q}'$ shown in Fig.1(b)) in the domain Ω' can be constructed directly by mapping the flow pattern $\boldsymbol{q}$ in Ω with the help of the mapping $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$. By using the geometry-material equivalency, the necessary material distribution to realize this heat flow pattern in the domain Ω' is given by Eq.(8). Therefore, if we want to control a heat flow pattern in a given domain, we can just deform its geometry so that the $\boldsymbol{q}'$ follows the designed pattern (we realize a spatial mapping $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$), and Eq.(8) tells directly the material necessary to realize such a function. For some simple cases, such as spherical cloak[15], concentrator[16] and rotator[17], analytical mappings can be easily found. However for more complicated devices, numerical methods have to be employed to establish the relation between the mapping and functionality, which will be explained in the next section.

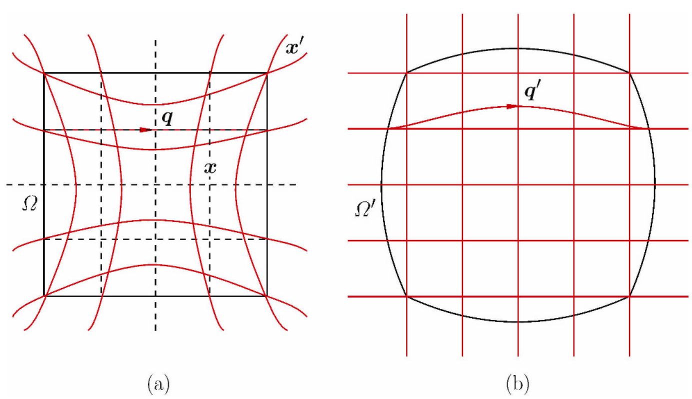

Fig. 1. Scheme of transformation method.

2.2. Construction of Transformation by Grid Generation Method

Once the function defined by a mapping is known, Eq.(8) gives automatically the necessary material for realizing such a function. So the key point to control heat flow is to construct the functional mapping for a designed device. Usually the desired function is only prescribed by a known mapping on certain boundaries, and the whole transformation enclosed by the boundary is left undetermined. Generally, the whole mapping can be constructed by solving the following boundary value problem:

$$f(\boldsymbol{x}(\boldsymbol{x}')) = 0 \tag{10a}$$

$$\boldsymbol{x}(\boldsymbol{x}')\,\big|_{\partial\Omega=\boldsymbol{\omega}'(\boldsymbol{x}')} = \boldsymbol{\omega}'(\boldsymbol{x}'), \quad \boldsymbol{x}(\boldsymbol{x}')\,\big|_{\partial\Omega=\boldsymbol{b}'(\boldsymbol{x}')} = \boldsymbol{b}(\boldsymbol{x}') \tag{10b}$$

where $\boldsymbol{x}'(\boldsymbol{x})$ denotes the original coordinate for a given point placed in the transformed physical space. This process can be realized through different methods, and the commonly used method is a linear interpolation from the boundary values, proposed originally for designing spherical and cylindrical cloaks[7]. However, for a more complex geometry, especially when the boundaries ω' and b' cannot be expressed in an analytical form, an alternative method should be proposed.

In planar grid generation[12], Eq.(10) describes a typical grid generation problem and Eq.(10a) is called the grid generator. Among various forms of the available generators, the widely used one is Winslow generator

$$\nabla^2_{\boldsymbol{x}'}(\boldsymbol{x}(\boldsymbol{x}')) = 0 \tag{11}$$

This generator is a classical elliptic partial differential equation, i.e. Laplace's equation, and Dirichlet boundary conditions are given as those in Eq.(10b). In this context, the mapping construction has been transformed into a boundary value problem of Laplace's equation, and it can be easily solved by a numerical technique, e.g. the finite element method. This method is first proposed by Hu et al.[10] for the design of electromagnetic devices with arbitrary shapes. After getting the mapping $\boldsymbol{x}(\boldsymbol{x}')$ by solving Eq.(11) with the boundary condition Eq.(10b), the Jacobian matrix of the mapping, Eq.(7), can be obtained by an inverse operation $\boldsymbol{A} = (\partial\boldsymbol{x}(\boldsymbol{x}')/\partial\boldsymbol{x}')^{-1}$, and the designed materials can then be obtained from Eq.(8). Another advantage of this method is that the material property and distribution in a designed region can be controlled and optimized. For example, it is proved that the grid generated by Winslow generator is the smoothest one[21], and a quasi-isotropic material design can be obtained if the boundary is allowed to slide, which generates a quasi-conformal mapping[11].

III. NUMERICAL APPLICATIONS

In this section, two examples will be given to illustrate the proposed method. In the first example, we will show how to create a temperature pattern by properly designing the conductivity distribution for a steady heat transfer problem. In the second example, we will illustrate how to produce a desired temperature pattern by managing heat source and material distribution for a transient heat conduction problem.

3.1. Temperature Pattern Control

A rectangle domain is prescribed with the following temperature boundary conditions $T^{\mathrm{H}} = 393$ K, $T^{\mathrm{L}} = 293$ K on its left and right edges, respectively, while the upper and lower edges are insulated, which is shown in Fig.2(a). If the domain is filled by a homogeneous and isotropic material with conductivity $k_0 = 300$ W/(m· K), the temperature distribution would be linear from left to right, as shown in Fig.2(b). Now we want the temperature pattern in an arbitrary region B in the domain Ω (Fig.2(a)) to be shifted to B' in the domain Ω' (Fig.2(c)) with the domain boundary conditions unchanged, and the question is how to design the material in the domain Ω'. To this end, we consider the homogeneous and isotropic domain shown in Fig.2(a) as the virtual space, and place the region B by a translation and rotation to B' (prescribing a spatial mapping), forming therefore a transformed domain Ω'. If this transformation can be evaluated, i.e. $\boldsymbol{A}(\boldsymbol{x}')$ in the transformed domain Ω', then the required conductivity distribution can be obtained directly by Eq.(8).

Mathematically, the boundary conditions of the spatial mapping can be described as follows:

$$\boldsymbol{x}(\omega') = \omega, \quad \boldsymbol{x}(b') = b \tag{12}$$

Or specifically on the outer boundary of Ω'

$$x = x', \quad y = y' \tag{13}$$

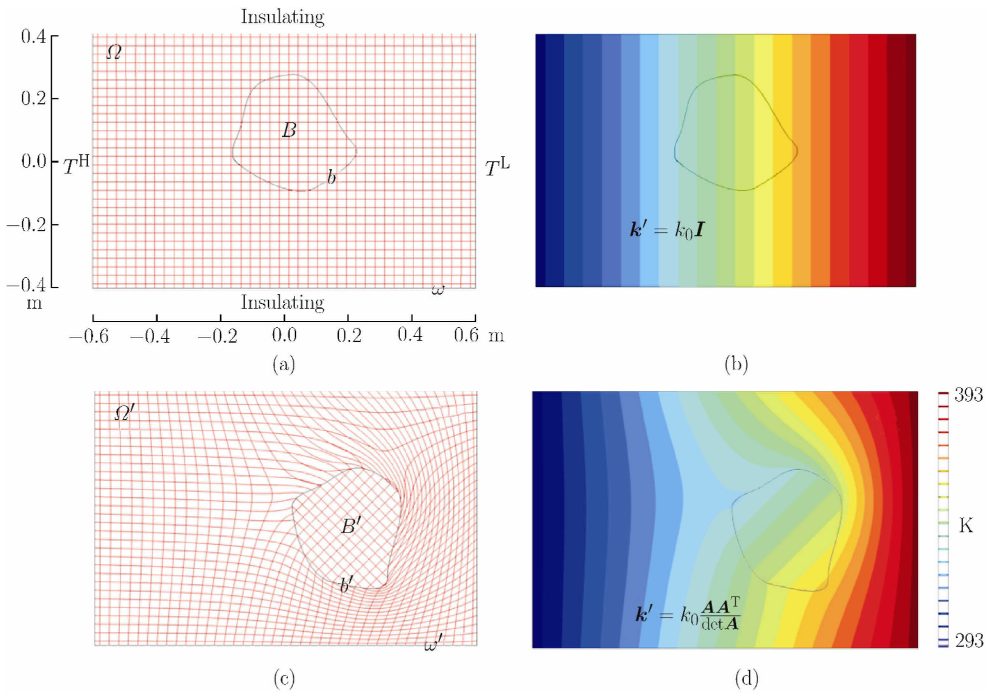

Fig. 2. Temperature pattern control for steady heat conduction: (a) Problem in a virtual space; (b) Temperature pattern in the virtual space; (c) Transformation constructed by grid generation method; (d) The designed temperature pattern.

and on the boundary of the region B', we have

$$\begin{aligned} x &= (x' - x_0)\cos(\varphi) + (y' - y_0)\sin(\varphi) \\ y &= -(x' - x_0)\sin(\varphi) + (y' - y_0)\cos(\varphi) \end{aligned} \tag{14}$$

where (x_0, y_0) are translation distances in the x and y directions, and φ is the angle of rotation. In this example, we set $(x_0, y_0) = (0.5, 0)$, and $\varphi = -\pi/4$. With the transformation of the boundary condition (Eqs.(13) and (14)), the whole transformation in the domain Ω' can be generated numerically by Winslow generator, i.e. Eq.(11), and the PDE module provided by the COMSOL package can be used for such a purpose. The solution $\boldsymbol{x}(\boldsymbol{x}')$ is just the original coordinates for a point $\boldsymbol{x}$ in the domain Ω. Figure 2(c) illustrates the deformed mesh lines from the original flat space Ω, and the line path from left to right implies the direction of the heat flux.

With the obtained solution $\boldsymbol{x}(\boldsymbol{x}')$, the designed conductivity $\boldsymbol{k}'$ is evaluated by Eq.(8). To validate the design, we calculate the temperature pattern with the obtained conductivity $\boldsymbol{k}'$ for a steady heat transfer problem, and the result is shown in Fig.2(d). By comparison with the temperature pattern in Fig.2(b), it is found that the temperature pattern in the region B' are indeed translated and rotated as desired.

3.2. Heat Source Transformation

In the second example, we will consider a transient heat transfer problem: an irregular domain Ω' consisting of three regions with a temperature $T = 293$ K prescribed on its right edge and the others remaining insulated, as shown in Fig.3(a). The initial temperature of the whole domain is set to be $T_{\mathrm{ini}} = 293$ K. If the region $1'$ is suddenly imposed and maintained with a homogeneous heat source $S_0 = 10^8$ W/m^3, the temperature pattern of the regions $2'$ and $3'$ at time $t = 1$ s is illustrated in Fig.3(a) if they are filled with a homogeneous and isotropic media with heat conductivity $k_0 = 300$ W/(m· K) and heat capacity $C_0 = 300$ J/K . Now, if we want a linear temperature distribution from left to right in the region $3'$ at any instant, the question is how to design the source in the region $1'$ and the material property in the region $2'$ in order to achieve this goal. To proceed, we consider an auxiliary problem: a rectangle domain Ω is filled with a homogeneous material ($k_0 = 300$ W/(m·K)), $C_0 = 300$ (J/K) and a homogeneous heat source ($S_0 = 10^8$ W/m^3) is imposed suddenly in the region 1, the boundary conditions remain the same as those in the domain Ω', as shown in Fig.3(b), which forms a transient heat conduction problem in the virtual space Ω. Now, we perform a spatial mapping from Ω to Ω' which keeps the region 3 unchanged, and the regions 1 and 2 in Ω are mapped to the regions $1'$ and $2'$ in Ω'. Winslow generator will again provide the transformation in the whole domain, therefore the necessary material and heat source to achieve a linear temperature pattern are determined by Eq.(8). To be more specific, we consider a mapping with the following boundary conditions:

$$x = -0.051, \quad y = \frac{0.2y'}{0.5}, \quad \text{on} \quad \partial\omega_1 \tag{15a}$$

$$x = x', \quad y = 0.5 \quad \text{on} \quad \partial\omega_2 \tag{15b}$$

$$x = x', \quad y = y' \quad \text{on} \quad \partial\omega_3 \tag{15c}$$

$$x = x', \quad y = -0.5 \quad \text{on} \quad \partial\omega_4 \tag{15d}$$

where ω_i, shown in Fig.3(c), represents different sections of the boundary. Note the choice of these boundary conditions is not unique, and Eq.(15) just provides a possible option. With Eqs.(11) and (15), the transformation can be easily obtained numerically by the PDE module of COMSOL package, which is given in Fig.3(c). Once the mapping is evaluated, the necessary conductivity and capacity of the material in the region $2'$, together with the heat source in the region $1'$ can be calculated from Eq.(8), which are now position-dependent. To validate the design, the temperature profiles are calculated with the obtained heat source S' and material parameters ($\boldsymbol{k}'$ and C'), and the result is shown in Fig.3(b) for $t = 1$ s. Indeed, the temperature pattern is linear in the region $3'$ at any instant, as designed.

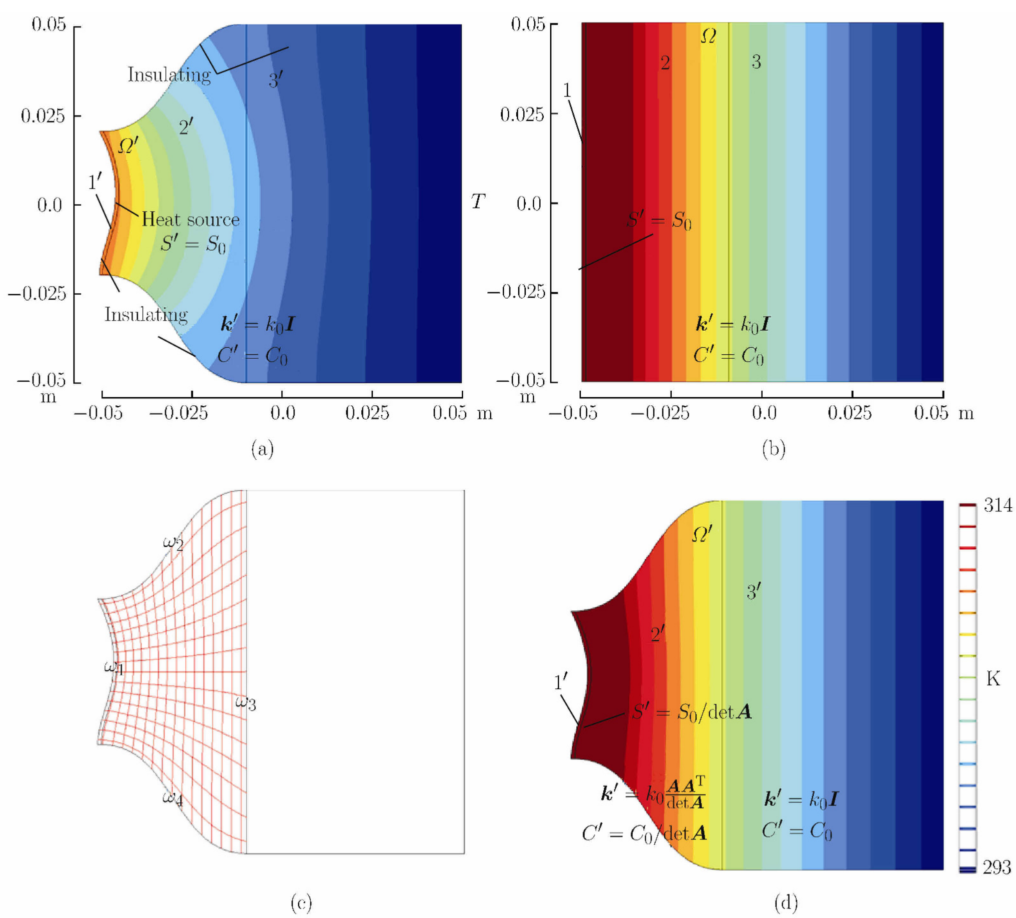

Fig. 3. Temperature pattern control by heat source transformation: (a) Problem in a physical space and the temperature pattern with a homogeneous heat source and material at $t = 1$ s; (b) Temperature pattern in a virtual space; (c) Transformation constructed by grid generation method; (d) Designed temperature pattern at $t = 1$ s.

IV. CONCLUSIONS

By combing the transformation method and grid generation technique, we proposed a general method to design material and heat source spatial distributions for heat flux or temperature control. The proposed method can deal with heat flux control problems in a domain of an arbitrary shape with the help of mapping generated by a numerical method. Two numerical examples are also provided to illustrate the capacity for temperature pattern control in the case of steady and transient heat conduction problems.

References

[1] Alifanov,O., Inverse Heat Transfer Problems. 1994, Springer, Berlin.
[2] Estorf,M., Image based heating rate calculation from thermographic data considering lateral heat conduction. *International Journal of Heat and Mass Transfer*, 2006, 49: 2545-2556.
[3] Woodfield,P.L., Monde,M. and Mitsutake,Y., Improved analytical solution for inverse heat conduction problems on thermally thick and semi-infinite solids. *International Journal of Heat and Mass Transfer*, 2006, 49: 2864-2876.
[4] Cossali,G.E., A Fourier transform based data reduction method for the evaluation of the local convective heat transfer coefficient. *International Journal of Heat and Mass Transfer*, 2004, 47: 21-30.
[5] Taler,J., A new space marching method for solving inverse heat conduction problems. *Forsch Ingenieurwes*, 1999, 64: 296-306.
[6] Chang,Z., Hu,J. and Hu,G.K., Transformation method and wave control. *Acta Mechanica Sinica*, 2010, 26: 889-898.
[7] Pendry,J.B., Schurig,D. and Smith,D.R., Controlling electromagnetic fields. *Science*, 2006, 312: 1780-1782.
[8] Leonhardt,U., Optical conformal mapping. *Science*, 2006, 312: 1777-1780.
[9] Chen,H. and Chan,C.T., Acoustic cloaking in three dimensions using acoustic metamaterials. *Applied Physics Letter*, 2007, 91: 183518.
[10] Hu,J., Chang,Z. and Hu,G.K., Approximate method for controlling solid elastic waves by transformation media, *Physical Review B*, 2011, 84: 201101(R).
[11] Chang,Z., et al., Transformation ray method: Controlling high frequency elastic waves. *Journal of the Acoustical Society of America*, 2012, 132(4): 2942-2945.
[12] Farhat,M., et al., Broadband cylindrical acoustic cloak for linear surface waves in a fluid. *Physical Review Letters*, 2008, 101: 134501.
[13] Zhang,S., et al., Cloaking of matter waves. *Physical Review Letters*, 2008, 100: 123002.
[14] Greenleaf,A., Lassas,M. and Uhlmann,G., On non-uniqueness for Calderón's inverse problem. *Mathematical Research Letters*, 2003, 10: 685-693.
[15] Fan,C.Z., Gao,Y. and Huang,J.P., Shaped graded materials with an apparent negative thermal conductivity. *Applied Physics Letter*, 2008, 92: 251907.
[16] Guenneau,S., Amra,C. and Veynante,D., Transformation thermodynamics: cloaking and concentrating heat flux. *Optics Express*, 2012, 20(7): 8207-8218.
[17] Narayana,S. and Sato,Y., Heat flux manipulation with engineered thermal materials. *Physical Review Letters*, 2012, 108: 214303.
[18] Schittny,R.,et al., Experiments on transformation thermodynamics: molding the flow of heat. *Physical Review Letters*, 2013, 110: 195901.
[19] Hu,J., Zhou,X.M. and Hu,G.K., Design method for electromagnetic cloak with arbitrary shapes based on Laplace's equation. *Optics Express*, 2009, 17(3): 1308-1320.
[20] Chang,Z., et al., Design method for quasi-isotropic transformation materials based on inverse Laplace's equation with sliding boundaries. *Optics Express*, 2010, 18(6): 6089-6096.
[21] Knupp,P. and Steinberg,S., Fundamentals of Grid Generation. 1994, CRC Press, Boca Raton.